Oscillatory Control for Constant-Speed Unicycle-Type Vehicles

Emmett Lalish

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Emmett Lalish

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Committee Members:

______________________________
Kristi A. Morgansen

______________________________
Rolf Rysdyk

Date: ____________________________
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Abstract

Oscillatory Control for Constant-Speed Unicycle-Type Vehicles

Emmett Lalish

Chair of the Supervisory Committee:
Professor Kristi Morgansen
Aeronautics & Astronautics

This thesis addresses a method for relating the dynamics of a constant-speed planar unicycle vehicle to a less constrained dynamical model. This model is appropriate for Unmanned Aerial Vehicles (UAVs), as their airspeed range is often significantly constrained. An oscillatory regulator is examined which controls a limit-cycle behavior. The regulator is based on the idea of defining a center of oscillation (CO) and using a parameterized oscillatory input to produce desirable CO dynamics. This definition allows the CO to be modeled in its own right, and its dynamics are subject to fewer constraints than those of the original vehicle. The CO can then be controlled using any of a variety of outer-loop controllers. Two outer-loop controllers are developed for the purposes of point stabilization and target tracking, respectively. Results are shown in simulation to validate the modeling techniques employed.
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Chapter 1

INTRODUCTION

Coordinated control of Unmanned Aerial Vehicles (UAVs) is a relatively new and quickly growing field of study. It presents unique challenges to the control designer because of the underactuated nature of aircraft as well as the physical limitations on the controls. The difference between this type of vehicle and variable speed vehicles is highlighted by the fact that the simple task of maintaining a fixed position is not possible for fixed-wing aircraft.

In fixed-altitude applications where UAVs have a significantly restricted flight envelope, the vehicles are often modeled as planar constant-speed unicycles (i.e. particles whose heading can be controlled). A number of techniques have been developed for controlling variable speed vehicles (e.g. [10, 14]), but methods for constant-speed vehicles are fewer (e.g. [13] for path following and [11, 4] for trajectory tracking). Extensions of these ideas to multivehicle systems can be found in [11, 5, 7], but in these applications either the number of vehicles or the allowable types of trajectories are limited. This thesis provides a method for control which does not suffer from these limitations.

Oscillatory control of nonlinear systems has been studied in a number of instances using either classical averaging theory [12, 3, 15] or methods using differential geometry [10, 14, 9]. However, one of the conditions of averaging theory is that the oscillatory terms must have constant amplitude over whole periods, allowing, at best, discrete-time state feedback. The tools for state feedback from differential geometry, on the other hand, do allow state-valued amplitude modulation of oscillations, but an underlying assumption for existing results is that the systems are small time locally (possibly only configuration) controllable. The work here differs from these techniques in two primary aspects: the system is not small time locally controllable (although it is controllable), and the amplitude of the oscillatory terms will be allowed to vary freely as a function of the state.

The methods developed in this paper are based on the idea of a “Center of Oscillation” (CO) which is an approximation of the actual dynamics, but not a true average, and has fewer constraints
than the actual system. In particular, while the original dynamics are controllable but not small time locally controllable (STLC), the CO dynamics are STLC, thus allowing the use of a broader range of techniques for the CO.

Two outer-loop controllers are then designed to make the CO track a target. One of these controllers is similar in derivation to [1], but in comparison the controller here admits moving targets. A virtual leader is used to accomplish this task, and is related in functionality to methods in [13] and [8]. Coordinated control is then achieved by coupling several such systems via particular choices of heading phase separation. Additionally, methods are presented for faster interception of a target and for compensating for the effect of wind.

This thesis is organized as follows. In Chapter 2 some mathematical preliminaries are reviewed and the control objective and the basic vehicle model are given. The oscillatory inputs as well as the CO and its model are described in Chapter 3. Outer-loop controllers for point stabilization (or parking) and target tracking are shown in Chapter 4. Concluding remarks and topics of current and future work are given in Chapter 5. Appendix A contains a basic stability proof that is referenced in Chapter 4.
Chapter 2
MATHEMATICAL PRELIMINARIES AND PROBLEM FORMULATION

2.1 Nonlinear Controllability

For a system to be controllable, it must be possible to get from any point in the state space domain to any other point in finite time. Formally, given:

\[ \dot{x} = f(x, u) \quad \text{and} \quad x_0, x_1 \in \mathbb{R}^n, \]

(2.1)

the system is controllable if and only if there exists a \( T > 0 \) and a \( u(t) \in [0, T] \) such that

\[ x(0) = x_0 \quad \text{and} \quad x(T) = x_1. \]

(2.2)

Even underactuated systems, like the constant-speed unicycle studied here, can be controllable in the nonlinear sense.

A stricter definition of controllability for nonlinear systems is known as small-time locally controllable (STLC). To have this property a system must be able to get to any local point (arbitrarily small distance, in any direction in the state space domain) in arbitrarily small time. Therefore this definition of controllability mimics more closely the standard linear definition, which says at any point the system can be pushed in any direction within the state space. The constant-speed unicycle studied here is not STLC because it cannot reverse direction.

2.2 Lyapunov Stability

A powerful nonlinear stability tool is Lyapunov theory. If a function can be found which satisfies the criteria in Theorem 1, this function known as a Lyapunov function and becomes a sufficient condition to ensure stability of a nonlinear system. However, no general rules exist for constructing a Lyapunov function, nor even ensuring one exists. One useful statement of this theory is the following theorem.
**Theorem 1** (Barbashin-Krasovskii theorem [6]). Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function such that

\[ V(0) = 0 \quad \text{and} \quad V(x) > 0, \quad \forall \; x \neq 0 \quad (2.3) \]

\[ ||x|| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty \quad (2.4) \]

\[ \dot{V}(x) < 0, \quad \forall \; x \neq 0 \quad (2.5) \]

then $x = 0$ is globally asymptotically stable.

A function $V$ satisfying (2.3) is said to be positive definite. Likewise, if (2.5) is satisfied, $\dot{V}$ is said to be negative definite. $V$ is said to be radially unbounded if (2.4) is satisfied.

### 2.3 Time Scale Separation

A useful tool for simplifying the analysis of control systems is the concept of time scale separation. This concept arises in many forms, but the fundamental idea is that slowly-varying portions of a system will not strongly influence faster dynamics. Therefore to analyze the fast dynamics, the slowly-varying part of the system can be assumed constant without introducing much error. A formalization of this concept is the following lemma.

**Theorem 2** (Riemann-Lebesgue Lemma [2]). For any function $h(\xi)$ integrable on $[a, b]$, any real value $\phi$, and any $\epsilon > 0$, there exists $\omega_0$ such that

\[ \left| \int_a^b h(\xi) \cos(\omega \xi + \phi) d\xi \right| < \epsilon, \quad \forall \; \omega \geq \omega_0. \quad (2.6) \]

In this formulation, $h(\xi)$ is the slowly-varying part of the system while $\omega$ represents the frequency of the fast dynamics. The statement can be interpreted as saying that so long as one set of dynamics is sufficiently faster (or time scale separated) from the other, then their interaction will be small. No general rule exists that says how much separation is enough, because this is completely problem-dependent.
2.4 Problem Formulation

Although complete aircraft dynamics are quite complex, the essential components for level cruise flight can be captured by the model of a constant-speed unicycle:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
v \cos \psi \\
v \sin \psi \\
u
\end{bmatrix},
\]

where \(v\) is the (constant) speed of the vehicle, and \(u\) is the control input (heading rate). This model is challenging to control because the vehicle cannot reverse direction (nor move directly sideways), and so it is not STLC. The purpose of this thesis is to develop input and state maps that relate this system to a system that is STLC and has dynamics

\[
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\bar{y}} \\
\dot{\bar{\psi}} \\
\dot{\bar{v}}
\end{bmatrix} =
\begin{bmatrix}
\bar{v} \cos \bar{\psi} - u_1 \sin \bar{\psi} \\
\bar{v} \sin \bar{\psi} + u_1 \cos \bar{\psi} \\
u_2 \\
u_3
\end{bmatrix},
\]

where \(u_1, u_2,\) and \(u_3\) are control inputs, and \(u_1\) will be designed to only be available when \(\bar{v}\) is small. Therefore at high speed this model will be that of a unicycle with acceleration input:

\[
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\bar{y}} \\
\dot{\bar{\psi}} \\
\dot{\bar{v}}
\end{bmatrix} =
\begin{bmatrix}
\bar{v} \cos \bar{\psi} \\
\bar{v} \sin \bar{\psi} \\
u_2 \\
u_3
\end{bmatrix}.
\]

Note that this system is STLC and has an equilibrium at the origin. Specifically, at low speed and with a transformation to body-fixed coordinates, the model becomes that of a fully controllable linear system:

\[
\begin{bmatrix}
\dot{\bar{x}}_b \\
\dot{\bar{y}}_b \\
\dot{\bar{\psi}} \\
\dot{\bar{v}}
\end{bmatrix} =
\begin{bmatrix}
\bar{v} \\
u_1 \\
u_2 \\
u_3
\end{bmatrix}.
\]
Once input and state maps are derived, this thesis will show that the distance between the positions of the vehicle (2.7) and this average system is a bounded, symmetric limit cycle where at each half period $\bar{x} = x$ and $\bar{y} = y$. 
Chapter 3
OSCILLATORY CONTROL

The limiting condition for (2.7) that precludes STLC is the lack of arbitrary control of translational velocity. In [7] the average velocity of three such vehicles was controlled by a coupled heading-rate oscillation. Motivated by that approach, consider the system (2.7) with heading rate parameterized by amplitude and fixed frequency:

\[ u(t) = A(t) \cos(\omega t). \]  

(3.1)

When \( A = 0 \) the vehicle travels at full speed, \( v \), and when \( A > 0 \) is constant, the vehicle effectively slows its forward progress by switchbacking across its path. Note that as \( A \) increases, the effective vehicle progress slows down.

From the standpoint of control and trajectory tracking, one is now more interested in the mean path of the vehicle, rather than the vehicle’s actual trajectory. As will be shown for a vehicle model (2.7) with input (3.1), this mean path can be modeled by (2.8) which will be referred to as the center of oscillation (CO). In this model, \( \bar{v} \) is the effective speed of net progress of the vehicle, \( \bar{\psi} \) is the orientation of this effective motion, and \((\bar{x}, \bar{y})\) denote the position on the mean path and will equal \((x, y)\) at half period intervals. The relationship between the vehicle and the CO (for \( \bar{\psi} = 0 \)) is shown graphically in Fig. 3.1 for three different values of \( \bar{v} \). The exact relationship between \( A \) and \( \bar{v} \) is discussed below.

For generality, control of the phase and frequency of the oscillation is also desirable, as such control allows multiple vehicles to be synchronized. To this end, the control (3.1) can be generalized to the form

\[ u(t) = A \cos(\phi(t)), \]  

(3.2)

where

\[ \dot{\phi}(t) = \omega(t), \]  

(3.3)
and $\omega$ is a control variable. Now consider a simple first-order controller for $\phi$:

$$\omega(t) = \omega_0 - k_\omega (\phi(t) - \omega_0 t - \phi_0), \quad (3.4)$$

where $\omega_0$ is the trim frequency, $k_\omega$ is a positive gain, and $\phi_0$ is the commanded reference phase. The phase error, $(\phi - \omega_0 t - \phi_0)$, is wrapped to the domain $(-\pi, \pi]$ and will converge to zero with this controller. To facilitate tractable analysis in the following work, $\omega$ will be chosen to have limited range and small rate. Choosing $k_\omega$ such that $k_\omega \pi << \omega_0$ ensures these constraints are met.

In the preceding discussion, only constant velocity of the CO was achievable. To enable a full range of motion, let

$$u = A(t) \cos(\phi(t)) + B(t) + C(t) \sin(2\phi(t)). \quad (3.5)$$

In this formulation, $A$, $B$, and $C$ are each time-varying inputs. However, in modeling the CO dynamics, $A$, $B$, and $C$ are assumed to be slowly-varying to simplify analysis. Using the Riemann-Lebesgue Lemma (Theorem 2), one can choose the gains such that the frequencies of the outer-loop controller are significantly slower than $\omega$, and so approximating the inputs as having zero derivative is reasonable. The exact bound on “significant” has yet to be determined (an order of magnitude is conservative), but the simulations shown in Chapter 4 demonstrate appropriate choices of gains.
With slowly-varying inputs, the first and third terms of (3.5) average to approximately zero. Therefore the bias, \( B \), has primary effect on the overall turning of the system, or more formally, \( \dot{\psi} \approx B \). This effect is demonstrated in Fig. 3.2. With \( \bar{v} \) determined by \( A \), and \( \bar{\psi} \) determined by \( B \), the CO now appears to be a variable-speed unicycle similar to (2.9):

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\psi} \end{bmatrix} \approx \begin{bmatrix} \bar{v} \cos \bar{\psi} \\ \bar{v} \sin \bar{\psi} \\ B \end{bmatrix},
\]

which is a vast improvement in maneuverability over the constant-speed nature of the original vehicle.

The effect of the \( C \) input on the CO dynamics is less straightforward. First, it only has a meaningful effect when \( \bar{v} \) is near zero, because it is designed to take advantage of the nonlinear coupling between the \( A \) term and the \( C \) term. As shown in the lower left of Fig. 3.2, the effect of \( C \) is to give the system (3.6) velocity in a direction perpendicular to the heading of the CO. The CO model emulates (2.8):

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\psi} \end{bmatrix} \approx \begin{bmatrix} \bar{v} \cos \bar{\psi} - rC \sin \bar{\psi} \\ \bar{v} \sin \bar{\psi} + rC \cos \bar{\psi} \\ B \end{bmatrix},
\]

which is linear and controllable when transformed to body-fixed coordinates, as in (2.10). The proportionality constant, \( r \), relates \( C \) to \( u_1 \) from (2.8).

In order to find relations between the inputs and the dynamics which are independent of \( v \) and \( \omega \), let the following parameters be constructed. First are normalized forms of \( \bar{v} \) and \( A \):

\[
\nu \equiv \frac{\bar{v}}{v} \quad \text{and} \quad \eta \equiv \frac{A}{\omega},
\]

where \( \nu < 1 \) is termed the speed fraction and \( \eta \) the oscillation parameter. Last is a scale parameter,

\[
\Upsilon \equiv \frac{v}{\omega},
\]

which has units of length and is used to normalize any distances in the system dynamics. These parameters will be used extensively in the following analysis and allow the control to be written as

\[
u = \eta \omega \cos(\phi) + B + C \sin(2\phi).
\]
Figure 3.2: Comparison of $B$ and $C$ inputs. On the lower left is $\dot{v} = 0$, $B = 0$ and $C = 0.1$ rad/s, while the other trajectory is $\dot{v} = 0.5$, $B = 0.1$ rad/s and $C = 0$. $v = 1$.

### 3.1 Mapping the Vehicle to the Center of Oscillation

Each CO state is coupled to the current state of the vehicle and input, so that the CO can be used to construct control for (2.7). In order to formulate this mapping in a tractable way, approximate steady state behavior is assumed for the CO system, meaning the CO moves in a constant direction with a constant speed. This assumption implies $B \approx C \approx 0$, and $\eta$ and $\omega$ are constant (or slowly-varying).

Beginning with the definition of vehicle heading, and substituting (3.10) into the definition of $\psi$ from (2.7) yields

$$\psi(t) = \int_0^t u(\tau)d\tau = \int_0^t \eta \omega \cos(\phi(\tau))d\tau. \quad (3.11)$$

Performing the integration while keeping in mind that $\dot{\phi} = \omega \approx \omega_0$ gives

$$\psi \approx \eta \sin(\phi) + \tilde{\psi}, \quad (3.12)$$

where $\tilde{\psi}$ is the integration constant for $\psi$ as well as the heading of the CO. Thus

$$\tilde{\psi} \approx \psi - \eta \sin(\phi) \quad (3.13)$$
defines the heading for the CO kinematics, such that in steady-state, $\psi$ oscillates about the constant $\bar{\psi}$.

Returning to the relationship between $\eta$ and $\nu$, determining the speed of the CO requires integrating the vehicle’s motion over a whole cycle, then dividing by the period. Because the vehicle’s net motion is in the direction of $\bar{\psi}$, one can (without loss of generality) simply integrate $\dot{x}$ with $\bar{\psi} = 0$ to obtain

$$\bar{v} = \frac{1}{T} \int_0^T \nu \cos(\psi) dt,$$

(3.14)

where $T = 2\pi/\omega$. Substituting (3.12) for $\psi$ and changing the integration variable from time to phase yields

$$\nu = \frac{1}{2\pi} \int_0^{2\pi} \cos(\eta \sin(\phi)) d\phi,$$

(3.15)

whose solution is a Bessel function of the first kind, of order zero:

$$\nu = J_0(\eta).$$

(3.16)

The function, $J_0$, is shown in Fig. 3.3 for reference. The value $\nu = 0$ occurs at $\eta \approx 2.40$, which means that for this value the vehicle will trace a figure-eight that will not move (see Fig. 3.1). This state corresponds to the vehicle turning back and forth exactly enough to close its path. In this work the domain of $J_0$ is restricted to $\eta \in [0, 3]$, resulting in the range $\nu \in [-0.26, 1]$. Over this domain, $J_0$ is invertible, and negative $\nu$ values result in the ability for the CO to reverse.

To couple the position of the CO to the vehicle, first construct a relative position vector between the CO and the vehicle, referenced to the CO’s heading:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos(\bar{\psi}) & \sin(\bar{\psi}) \\ -\sin(\bar{\psi}) & \cos(\bar{\psi}) \end{bmatrix} \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix}.$$

(3.17)

With the controls (3.10), both $\hat{x}$ and $\hat{y}$ will oscillate (see Fig. 3.1). The exact shape of these oscillations cannot be found in closed form, however numerical approximations can be made so that the CO position can be used for feedback. The $\hat{x}$ and $\hat{y}$ oscillations are found by running simulations of the vehicle over one period and comparing the result to the mean path definition of the CO. These oscillations (normalized by the scale parameter $\Upsilon$) can be characterized using a Discrete Fourier Transform over the range of $\eta$ values. A sample of this analysis for $\eta = 2.4$ is shown in Fig. 3.4.
This result shows that the first two dominant terms of the Fourier series make a good approximation of the true motion:

\[
\begin{align*}
\dot{x} & \approx \Upsilon \left[ g_2(\eta) \sin(2\phi) + g_4(\eta) \sin(4\phi) \right] \\
\dot{y} & \approx -\Upsilon \left[ h_1(\eta) \cos(\phi) + h_3(\eta) \cos(3\phi) \right],
\end{align*}
\]

(3.18)

where \( g_2, g_4, h_1 \) and \( h_3 \) are Fourier magnitudes as functions of \( \eta \). Numerical approximations of these functions are shown in Fig. 3.5. Finding the actual position of the CO, \((\bar{x}, \bar{y})\), is accomplished by inverting (3.17) and subtracting the vehicle’s coordinates. This method is used in Fig. 3.1 to plot the CO, and this approximation is shown to be quite accurate by the straightness of the CO trajectories.

### 3.2 Filtering

Because the states of the CO depend directly on \( \eta \), using \( \eta \) as a control input could easily result in a feedback resonance that would drive the system unstable. To avoid this problem, \( \eta \) is filtered by an...
Figure 3.4: Discrete Fourier Transform of $\hat{x}$ and $\hat{y}$ oscillations for $\eta = 2.4$.

Figure 3.5: Fourier series dimensionless magnitudes as functions of $\eta$. 
integrator. The new control input, \( a \), acts like an acceleration. The extended vehicle model is then

\[
\frac{d}{dt} \begin{bmatrix}
    x \\
    y \\
    \psi \\
    \phi \\
    \eta
\end{bmatrix} = \begin{bmatrix}
    v \cos \psi \\
    v \sin \psi \\
    \eta \omega \cos \phi + B + C \sin(2\phi) \\
    \omega \\
    a
\end{bmatrix},
\]  

(3.19)

where \( \omega \) is given by (3.4), and \( a \), \( B \) and \( C \) are inputs.

To simplify the relationship between \( a \) and \( \bar{v} \), a linear approximation of \( J_0 \) is used:

\[
q \approx -0.52.
\]  

(3.20)

Therefore the dynamics of \( \bar{v} \) can be represented as

\[
\dot{\bar{v}} = v \frac{d\nu}{d\eta} \approx v \frac{d\nu}{d\eta} \frac{d\eta}{dt} = vqa.
\]  

(3.21)

The error this approximation introduces into the model is that the actual acceleration will go to zero as \( \eta \to 0 \). Since the effective gain of the system decreases, it should not adversely affect stability, though it will somewhat reduce performance.

### 3.3 CO Model

The approximate CO model (3.7) is now

\[
\frac{d}{dt} \begin{bmatrix}
    \bar{x} \\
    \bar{y} \\
    \bar{\psi}
\end{bmatrix} \approx \begin{bmatrix}
    \bar{v} \cos \bar{\psi} - p\Upsilon C \sin \bar{\psi} \\
    \bar{v} \sin \bar{\psi} + p\Upsilon C \cos \bar{\psi} \\
    B \\
    qva
\end{bmatrix},
\]  

(3.22)

where in order to properly approximate (2.8), \( r \) has been replaced by \( p\Upsilon \), where \( p \approx 0.22 \) is a dimensionless proportionality constant (determined numerically), and \( \Upsilon \) scales \( C \) to the proper units. The inputs to this system are \( a \), \( B \), and \( C \).

This model leads to two modes of operation based on the limitations imposed on \( C \). If the CO needs to move quickly (more than about a tenth of the vehicle’s speed), then \( C \) will have little effect, and the model will behave as a variable-speed unicycle with acceleration and turning rate as inputs.
(a and $B$), similar to the system (2.9). If the CO needs to move slowly (e.g. for point stabilization), then the model will behave as a fully actuated vehicle with inputs of turning rate, acceleration along $\bar{\psi}$, and velocity perpendicular to $\bar{\psi}$ ($B$, $a$, and $C$, respectively), similar to the system (2.8).

As a graphical overview of the construction in the preceding sections, Fig. 3.6 shows a block diagram. This entire system is then approximated by the CO model (3.22), so that from the control design point of view, the system looks like Fig. 3.7. This model is based on approximations near the defined steady-state conditions, and is therefore only valid near this steady-state value. However, by assuring that the frequencies of any outer-loop controller used are sufficiently slower than $\omega$ (by Theorem 2) and that the controller used is robust, these approximations will be sufficiently accurate to ensure stability.

Requiring the controller to be relatively slow also means that the $B$ and $C$ terms of the input will be significantly smaller than the $\eta$ term. Therefore this system naturally accounts for heading rate limitations of the vehicle because $\eta < 3$, and so

$$|u| < \eta \omega_0 < 3\omega_0$$

for small $B$, $C$, and $k_\omega$. Therefore if a vehicle has a maximum heading rate, $\dot{\psi}_{\text{max}}$, the trim fre-
Figure 3.7: System block diagram used for designing outer-loop controller. The CO model masks the true dynamics shown in Fig. 3.6.

Frequency can be chosen as

$$\omega_0 < \frac{\dot{\psi}_{\text{max}}}{3}$$

(3.24)

to ensure no clipping of the commanded control input.
Chapter 4

OUTER-LOOP CONTROL

The method developed in the previous chapter allows any outer-loop controller to be wrapped around the CO model to achieve desired stability or tracking. Much literature already exists on controllers which are designed for variable-speed unicycles (like mobile robots), for everything from formation keeping to target tracking. Many of these controllers can be applied to this CO model (3.22), hence expanding their use to aircraft and other systems modeled by (2.7). Two example controllers are developed in the following sections. The first is a relatively simple algorithm for point-stabilization (sometimes referred to as the parking problem). The second (in Section 4.2) is an in-depth development of a target tracking algorithm.

While these controllers were designed specifically to integrate with the constraints of the oscillatory control inner-loop developed above, they are also applicable to vehicles that natively have dynamics similar to (2.8).

4.1 Point Stabilization

The outer-loop controller discussed here is designed to stabilize the CO to a desired position and orientation, a task commonly referred to as the parking problem when applied to nonholonomic vehicles. The equilibrium condition is for the CO to be stationary, which translates to the vehicle flying a static figure-eight pattern. This result can be thought of as a loiter condition for the aircraft. This controller is only meant to be used near the desired equilibrium, because it relies on assumptions of small forward and lateral velocity. For the tasks of approaching a waypoint from a distance or tracking a moving target, a better-performing controller is described in Section 4.2.

Theorem 3. The controls,

\[
\begin{bmatrix}
a \\
B \\
C
\end{bmatrix} = \begin{bmatrix}
-k_{x}(x \cos \psi + y \sin \psi) - k_{v} \bar{v} \\
-k_{\psi} \bar{v} \\
-k_{y} \bar{v} (y \cos \psi - x \sin \psi)
\end{bmatrix},
\]

(4.1)
applied to the dynamics (3.22) with all gains \((k_x, k_y, k_v, k_\psi)\) positive will globally asymptotically stabilize the CO to the origin.

**Proof.** Consider the following radially unbounded Lyapunov function,

\[
V = \frac{1}{2} \bar{x}^2 + \frac{1}{2} \bar{y}^2 + \frac{1}{2} \bar{\psi}^2 + \frac{1}{2k_x} \bar{v}^2,
\]

which has derivative

\[
\dot{V} = -k_y (\bar{x} \sin \bar{\psi} - \bar{y} \cos \bar{\psi})^2 - k_\psi \bar{\psi}^2 - \frac{k_v}{k_x} \bar{v}^2.
\]

\(\dot{V}\) is negative definite with respect to \(\bar{y}, \bar{\psi},\) and \(\bar{v},\) but only negative semi-definite with respect to \(\bar{x}.\) Therefore \(\bar{y}, \bar{\psi},\) and \(\bar{v}\) are globally asymptotically stable to the origin. To show that \(\bar{x}\) also converges to zero, consider the system dynamics with \(\bar{\psi} = 0:\)

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \bar{v} \\ -k_x \bar{x} - k_v \bar{v} \end{bmatrix}.
\]

Because this system is second-order linear with negative eigenvalues for all positive gains, \(\bar{x}\) will also be globally asymptotically stable to the origin (see Appendix A for proof). Therefore the whole system is globally asymptotically stable to the origin.

To demonstrate the use of this controller the following simulation uses \(v = 1\) and \(\omega = 1.\) The vehicle motion is shown in Fig. 4.1, and the corresponding controls are shown in Fig. 4.2. As can be seen, the vehicle stabilizes its CO to the proper position and orientation, corresponding to a figure-eight limit cycle motion of the actual vehicle.

### 4.2 Target Tracking

The goal of the control developed here is for the system (2.7) to track a moving target by way of converging the CO of the vehicle to the position of the target. The control inputs are subject to limitations of the form

\[
|a| \leq a_{\text{max}} \\
|B| \leq B_{\text{max}}
\]

in order to ensure that the CO model (3.22) represents the actual CO motion reasonably well. The target position is denoted \((x_t, y_t),\) and it moves with time-varying velocity, \(\bar{v}_t.\) This velocity vector
Figure 4.1: Single vehicle stabilizing to a fixed point. The vehicle is the red airplane while its CO is the red triangle. The vehicle’s path is denoted by a gray line (red for the last cycle), while its CO’s path is the dashed red line.

Figure 4.2: Control inputs corresponding to point stabilization (Fig. 4.1).
can be equivalently defined by orientation, $\psi_t$, and speed, $v_t$. To make tracking possible, the speed of the target is restricted to within the same range as the vehicle’s speed:

$$0 < v_t < v. \quad (4.6)$$

The algorithm presented here will prove asymptotic stability for tracking under the assumption of the target’s velocity being piecewise-constant. Additionally, if the vehicle is far from the target initially, the vehicle will approach it in an efficient way, allowing the methods here to address both target tracking and waypoint navigation (going from point to point). While this controller is tailored for use with the oscillatory control developed in Chapter 3, it can also be used on any vehicle with dynamics similar to (2.9). Therefore another promising application for this algorithm is the system in [8].

### 4.2.1 Conceptual Approach

Because this controller is designed to follow a moving target, generally $\nu$ will not be small, and so the input $C$ will not be particularly effective. Therefore in this controller, $C$ is set to zero, and the CO dynamics (3.22) simplify to (2.9):

$$\begin{bmatrix}
\ddot{x} \\
\ddot{y} \\
\ddot{\psi} \\
\ddot{v}
\end{bmatrix}
\approx
\begin{bmatrix}
\dot{v} \cos \bar{\psi} \\
\dot{v} \sin \bar{\psi} \\
B \\
qva
\end{bmatrix}. \quad (4.7)$$

The approach to target tracking considered here makes use of a polar coordinate system for control. However, because polar angles become ill-defined near the origin, locating the origin at the target would create a singularity at the point to which the CO is trying to stabilize. Instead, a virtual leader is constructed ahead of the target (in the direction $\psi_t$), leading by a distance $e^*$ to serve as the origin of the polar coordinate system:

$$\begin{align*}
x_l & \equiv x_t + e^* \cos \psi_t \\
y_l & \equiv y_t + e^* \sin \psi_t.
\end{align*} \quad (4.8)$$
The following state transformation (similar to the transformation used in [1]) is used to convert the Cartesian dynamics of the CO (4.7) into polar coordinates:

\[
\begin{align*}
e &= \sqrt{(x_l - \bar{x})^2 + (y_l - \bar{y})^2} \\
\alpha &= \tan^{-1}\left(\frac{y_l - \bar{y}}{x_l - \bar{x}}\right) - \bar{\psi} \\
\theta &= \alpha + \bar{\psi} - \psi_t.
\end{align*}
\] (4.9)

This coordinate system is depicted graphically in Fig. 4.3. Transforming the model (4.7) into the coordinates (4.9) yields

\[
\frac{d}{dt} \begin{bmatrix} e \\ \alpha \\ \theta \\ \bar{v} \end{bmatrix} = \begin{bmatrix} -\bar{v} \cos \alpha + v_t \cos \theta \\ -B + \bar{v} \sin \alpha e - v_t \sin \theta e \\ \bar{v} \sin \alpha e - v_t \sin \theta e \\ qva \end{bmatrix}.
\] (4.10)

This control algorithm was inspired by the natural stability exhibited by a trailer attached to a truck by a trailer hitch. Looking at this system within the reference frame defined in Fig. 4.3, the trailer takes the place of the CO, and the truck takes the place of the virtual leader. The mechanical linkage of the trailer hitch would force \( \alpha \) to be zero and \( e \) to be constant.

The CO model (4.10) must use the inputs \( a \) and \( B \) instead of a mechanical linkage. In order to derive similar dynamics to the trailer hitch system, the inputs will be used to drive \( \alpha \) to zero and \( e \)
to \( e^* \). When the target is moving with constant velocity, the CO will then stabilize to a distance \( e^* \) behind the virtual leader. Because the virtual leader is also \( e^* \) ahead of the target, the CO will be at exactly the same coordinates as the target, resulting in zero tracking error.

### 4.2.2 Tracking Algorithm

This section will prove stability for this controller mathematically. Bounds on the control gains will be derived relative to the saturation limits that ensure stability. The following proof uses a cascade of Lyapunov functions because a single Lyapunov function could not be found. The general concept of a cascade is that one Lyapunov function can be used to show that one state converges uniformly to zero (independent of the other state values). Then the dynamics can be simplified by substituting zero in for the converged state, and another Lyapunov function can be found. This process can be repeated as many times as necessary because by construction, each Lyapunov function does not affect the convergence of the one before it. Therefore, even though the last state appears to only converge locally, it actually retains global stability by the construction of the cascade.

**Theorem 4.** The controls

\[
\begin{bmatrix}
B \\
α \\
C
\end{bmatrix} = \begin{bmatrix}
k_α α + \bar{v} \frac{\sin α}{e} - v_t \frac{\sin θ}{e} \\
-k_e \frac{q}{q^*} (e^* - e) \cos α - k_v \frac{q}{q^*} (\bar{v} - v_t) \\
0
\end{bmatrix}
\]  

applied to dynamics (4.10) with all gains positive and \( v_t > 0 \), cause the CO to be globally asymptotically stable to the target’s position and velocity.

**Proof.** Choosing the radially unbounded Lyapunov function

\[
V_1 = \frac{1}{2} α^2
\]

yields the derivative

\[
\dot{V}_1 = α \left( -B + \bar{v} \frac{\sin α}{e} - v_t \frac{\sin θ}{e} \right).
\]

Choosing \( B \) to cancel terms gives

\[
B = k_α α + \bar{v} \frac{\sin α}{e} - v_t \frac{\sin θ}{e}.
\]
The derivative of the Lyapunov function is then negative definite:

$$\dot{V}_1 = -k_\alpha \alpha^2 \leq 0.$$  
(4.15)

Because the Lyapunov function and its derivative only include \( \alpha \), this Lyapunov function proves only that \( \alpha \to 0 \). Using (4.10), \( \dot{\theta} \) reduces to

$$\dot{\theta} = -v_t \sin \theta e.$$  
(4.16)

Therefore a second radially unbounded Lyapunov function can be constructed:

$$V_2 = \frac{1}{2} \theta^2.$$  
(4.17)

The derivative of this second function is

$$\dot{V}_2 = -\frac{v_t}{e} \theta \sin \theta \leq 0,$$  
(4.18)

which is negative definite for \( \theta \in (-\pi, \pi) \) (\( e \) and \( v_t \) are positive by definition). For the special case of \( \alpha = 0 \) and \( \theta = \pi \), the CO and target are approaching each other head on. This case is resolved by the fact that the CO will go through the singularity at the virtual leader, which will switch the angles to \( \alpha = \pi \) and \( \theta = 0 \). From there the system will stabilize as usual.

So far this analysis has shown \( \alpha \) and \( \theta \) are guaranteed to approach zero. Substituting this condition into \( \dot{e} \) in (4.10) leaves:

$$\dot{e} = v_t - \bar{v}.$$  
(4.19)

Similarly, \( a \) in (4.11) becomes:

$$a = -\frac{k_e}{q v}(e^* - e) - \frac{k_v}{q v}(\bar{v} - v_t).$$  
(4.20)

Changing coordinates to

$$x_1 = e^* - e$$
$$x_2 = \bar{v} - v_t$$  
(4.21)

yields the following linear ODE:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_e & -k_v \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$  
(4.22)
This system will be globally asymptotically stable to the origin (see Appendix A), which is:
\[
\begin{align*}
    e &= e^* \\
    \bar{v} &= v_t.
\end{align*}
\]
Because the velocities match, the angles are zero, and the CO lags the virtual leader by the same amount \(e^*\) that the virtual leader leads the target, the CO has converged to the same position and velocity as the target.

**Theorem 5.** Theorem 1 holds in the case where the controls are limited by (4.5).

**Proof.** This proof follows the previous one, but adds bounds on the gains to ensure stability is never broken. First, consider (4.13) and (4.14). \(\dot{V}_1\) will be ensured negative definite so long as \(B\) is at least large enough to cancel the indefinite terms:
\[
B_{\max} > \left| \frac{\bar{v}}{e} \sin \alpha - \frac{v_t}{e} \sin \theta \right|.
\]
Bounding this term by the worst case yields
\[
B_{\max} > \left| \frac{v}{e} \right| + \left| \frac{v}{e} \right| = 2\frac{v}{e},
\]
which gives a lower bound on \(e\) for stability. Choosing the leading distance such that
\[
e^* > \frac{2v}{B_{\max}}
\]
ensures that the equilibrium of the system will be stable by the same logic as in Theorem 1. The only zone where \(\alpha\) will not necessarily go to zero is when \(e < e^*\), however the controls (4.11) ensure in this case that \(\bar{v}\) will decelerate to less than \(v_t\), so the virtual leader will eventually pass the CO, causing \(e \geq e^*\). The second-order linear ODE (4.22) is also globally, asymptotically stable, provided
\[
\frac{k_i}{k_e} \geq \frac{v}{a_{\max}}.
\]
A Lyapunov proof of this is shown in Appendix A.

The rate at which \(\theta\) converges to zero is directly related to the speed of the target, \(v_t\). This can be seen intuitively from the trailer hitch concept this method is built around. Therefore if \(v_t = 0, \theta\)
will not converge to zero. To resolve this situation, the controller from Section 4.1 is used when $v_t$ is very small.

Simulation results are shown in Fig. 4.4 for a combination of this controller (4.11) and the controller from Section 4.1 (4.1). The simulation has three vehicles starting with identical initial conditions in the plane (but different altitudes). The target starts at a speed of 0.2 units/sec, then after 40 seconds turns and goes 0.4 units/sec, and finally after 40 more seconds, stops entirely. The only difference between the vehicles is that each is commanded to a different relative phase angle (to keep their target flyovers distributed evenly).

The first two legs of this simulation use the target tracking controller because the target is moving. However, when the target stops, the algorithm switches to point stabilization, which makes the CO reverse to compensate for overshoot and stabilize onto the stationary target.

The effects of unmodeled dynamics in the CO model are clearly visible in this simulation, because without perturbations, each vehicle’s CO would trace identical paths. The fact that the paths are different shows that relative phase affects the perturbation. However, despite these perturbations, the controller stabilizes to zero steady-state error in each of the three target velocity conditions.

The next simulation, shown in Fig. 4.5, is designed to exhibit the robustness of the modelling method developed here. The target tracking algorithm only ensures that the tracking error decreases to zero if the target’s velocity is constant. Therefore the target is shown at constant velocity, turning at 0.1 rad/s, and oscillating at frequency $\omega$ (all at speed 0.3). While non-constant target velocity does lead to steady-state error, the system still converges to a steady limit cycle which keeps the vehicle near the target.

The final simulation, shown in Fig. 4.6, demonstrates the effect of phase change and how it can be used in multi-vehicle coordination. Three vehicles were started with identical initial conditions, but were commanded to change their phase angles to be spaced evenly around the unit circle. As demonstrated, they converge successfully while tracking the same target (which moves at speed 0.2). This method can be useful for getting frequent flyovers of a target.
Figure 4.4: Three vehicle target tracking with $v = 1$ unit/sec, $\omega_0 = 1$ rad/sec, and identical initial conditions. Each vehicle’s path is denoted by a solid line, while its corresponding CO’s path is a dashed line. The target’s path is denoted by a black dashed line. The CO of each vehicle is shown as a triangle.
Figure 4.5: Single vehicle tracking a target. The target is denoted by the black triangle, and its path by the black dotted line. \( v = 1 \) unit/sec, \( \omega = 1 \) rad/sec and \( v_t = 0.3 \) unit/sec.
Figure 4.6: Three vehicles tracking a single target while changing relative phase. Each vehicle’s path is denoted by the solid line, and its corresponding CO’s path by the dotted line of the same color.
4.2.3 Interception

The controller described above causes $\alpha$ to go to zero, which means that the vehicle aims directly at the virtual leader, and when the vehicle is far away from the leader, the vehicle’s velocity will saturate at full speed. However, when far away, aiming directly toward a moving target is not the fastest way to approach it. To minimize the time taken to intercept, the optimal solution is to fly in a straight line such that the relative velocity vector between the vehicle and the target is aligned with the relative position vector between them. This method is a common practice in the literature on target interception.

This faster approach trajectory can be accomplished without changing the controller (4.11), but rather by adjusting the virtual leader’s distance, $L$. An easy way to calculate this new $L$ based on geometry is to draw a triangle, shown in Fig. 4.7. First, assume that the controls (4.11) ensure that $\alpha$ goes to zero and $\vec{v}$ goes to $v$. In order for the target and vehicle to reach the virtual leader at the same time:

$$\frac{e}{L} = \frac{v}{v_t}.$$  \hspace{1cm} (4.28)

Combining this result with the law of cosines yields

$$L = \frac{d}{\frac{v_t^2}{v^2}} \left( \cos \phi + \sqrt{\cos^2 \phi + \frac{v^2}{v_t^2} - 1} \right),$$  \hspace{1cm} (4.29)
where \( d \) and \( \phi \) are the target’s equivalent values of \( e \) and \( \theta \):

\[
\begin{align*}
    d &= \sqrt{(x_t - \bar{x})^2 + (y_t - \bar{y})^2} \\
    \phi &= \psi_t - \tan^{-1}\left(\frac{y_t - \bar{y}}{x_t - \bar{x}}\right). 
\end{align*}
\] (4.30)

This relationship is shown graphically in Fig. 4.7. Because \( d \sim L \sim e \), this triangle will shrink in time, but remain the same shape. Therefore none of the angles will change, and the virtual leader will remain at a fixed position while both the vehicle and target approach it.

The only necessary adjustment to the controller (4.11) is to set \( v_t \) to zero to reflect that the virtual leader is stationary instead of moving at the speed of the target:

\[
\begin{pmatrix}
    B \\
    a \\
    C
\end{pmatrix}
= \begin{pmatrix}
    k_\alpha \alpha + \frac{\bar{v} \sin \alpha}{e} \\
    -k_\alpha \frac{e^* - e}{\bar{v}} \cos \alpha - \frac{k_\alpha}{\bar{v}^2} \bar{v} \\
    0
\end{pmatrix}. 
\] (4.31)

The switchover between interception and stabilization happens automatically by projecting the virtual leader ahead of the target by a distance:

\[
L = \max(L, e^*). 
\] (4.32)

The system is still stable, because all of the Lyapunov arguments shown in Subsection 4.2.2 apply to the system both before and after the switching point.

The simulation in Fig. 4.8 shows the improvement of this direct interception method over the original target tracking algorithm. Snapshots of the vehicles, target, and virtual leaders are shown at regular intervals to reinforce that the blue vehicle’s virtual leader does not move during the target approach. Therefore the blue vehicle can fly in a straight line while the red vehicle has to go around as it follows the original virtual leader. Once the blue vehicle gets sufficiently close to the target, the two virtual leaders become one, and from there both vehicles stabilize in the usual way.

4.2.4 Wind Disturbance Rejection

To this point, modeling and control of this system has been performed while neglecting external disturbances. However, because this control is tailored to UAVs, wind is an important consideration,
Figure 4.8: Comparison of direct target interception (blue vehicle) to normal target tracking (red vehicle). For this simulation, \( v = 1 \) unit/sec, \( v_t = 0.7 \) unit/sec, \( \omega_0 = 1 \) rad/sec and \( e^* = 1 \) unit. Each vehicle’s path is denoted by the solid line, its CO by the dashed line, and the target by the black dashed line. The CO is denoted by a triangle and the virtual leader is a circle.
particularly when airspeed is on the same order as windspeed. Realistically, in order to approximately model an aircraft as a constant-speed unicycle, the vehicle speed should be airspeed rather than ground speed. This assumption means that wind enters the vehicle model in the following way:

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} v \cos \psi + v_{\text{wind},x} \\ v \sin \psi + v_{\text{wind},y} \\ u \end{bmatrix}
\]  

(4.33)

and similarly for the CO model:

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\psi} \\ \bar{\nu} \end{bmatrix} = \begin{bmatrix} \bar{v} \cos \bar{\psi} - p \bar{\Upsilon} C \sin \bar{\psi} + v_{\text{wind},\bar{x}} \\ \bar{v} \sin \bar{\psi} + p \bar{\Upsilon} C \cos \bar{\psi} + v_{\text{wind},\bar{y}} \\ B \\ q\nu a \end{bmatrix}
\]  

(4.34)

Accounting for such terms in the control laws developed here is straightforward. The velocity of the wind affects the system just as the velocity of the target does. Therefore instead of matching the velocity vector of the target \((\psi_t \text{ and } v_t)\), the CO must match the vector difference of the target’s velocity and the wind’s velocity:

\[
\vec{v}_{\text{total}} = \vec{v}_{\text{target}} - \vec{v}_{\text{wind}}.
\]  

(4.35)

This adjustment allows the controls derived above to work with piecewise-constant wind vectors, in the same way that they do for piecewise-constant target vectors. Further, not only are the commanded speed and heading affected, but the virtual leader position is also shifted, because it must now be projected along this total velocity vector, rather than along the target’s vector. The adjustment is in fact simply a change of coordinates, from the earth-fixed frame to the air-fixed frame. It has the effect of making the entire vehicle trajectory point far enough into the wind to negate wind effects. However, the admissible range of target velocities (4.6) must also change to account for the wind:

\[
|\vec{v}_{\text{total}}| < v.
\]  

(4.36)

Simulation results for the case of wind rejection are shown in Fig. 4.9. For the sake of comparison, this simulation has identical parameters to those used in Fig. 4.4, except that only one vehicle
is used (for clarity), and wind is present (denoted by the black arrow). In this example the wind vector is constant and of magnitude 0.2 units/sec.

The effect of wind on the system is clear in the way it skews the vehicle trajectory, but the CO still stabilizes to a zero steady-state error condition for each of the three target velocities. Also, when the target stops at the end of this simulation, the vehicle does not switch to the point stabilization controller from Section 4.1. The reason that the target tracker is used in this situation is because the total velocity vector is non-zero (hence the projected leader).
Figure 4.9: Single vehicle tracking a target with wind in the direction of the black arrow, with magnitude 0.2 unit/sec, $v = 1$ unit/sec, and $\omega_0 = 1$ rad/sec. The vehicle’s path is denoted by the red solid line, its CO by the red dashed line, and the target by the black dashed line. The CO is denoted by a triangle and the virtual leader is the red circle.
Chapter 5

CONCLUSION

In this thesis a method has been presented for relating the model of a constant-speed unicycle to that of an STLC system (the center of oscillation). This control method keeps the vehicle in a strict pattern around its CO, so that the maximum distance it will deviate is always known (3.18). In addition, these controls guarantee that the vehicle will fly directly over its CO two times per cycle. Multiple vehicles can also be coordinated through the use of relative phase control. The approximations that were made in this construction were validated by simulations.

Much work remains to be done with this method of control, but the results thus far are promising. The techniques presented here are particularly applicable to aircraft, where airspeeds are often restricted to the point that modeling them as constant-speed is quite reasonable (especially because an efficient cruise speed can be chosen). Additionally, the methods place natural limits on maximum turn rate. Therefore the frequency, $\omega_0$, can easily be chosen small enough that the controls never command a turning rate that is outside of the vehicle’s admissible range.

Additionally, outer-loop controllers were developed to track a stationary or moving target. The target tracking algorithm is useful not just for the constant-speed system studied here, but also for variable-speed unicycle models. This tracker is stable even in the presence of control saturation, has good target-approach performance, and can account for wind.

Topics for continuing and future work include testing this control technique on an actual aircraft, and thereby showing its robustness to real-world difficulties. Further, the results here will be made concrete with respect to classical averaging theory and differential geometric techniques. Finally, strict bounds will be placed on the allowable frequency of any outer-loop controller.
BIBLIOGRAPHY


Appendix A

STABILITY PROOF FOR A SECOND ORDER SYSTEM WITH SATURATED INPUT

Theorem 6. The controls

\[ a = -k_e x_1 - k_v x_2, \quad (A.1) \]

applied to dynamics,

\[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ a \end{bmatrix}, \quad (A.2) \]

cause the system to be globally, asymptotically stable to the origin for all positive gains, \( k_e \) and \( k_v \).

Proof. This is a well known result from linear system analysis, but the Lyapunov proof will be shown here so that it can be extended in the next theorem.

Choose the general quadratic Lyapunov function

\[ V = x_1^2 + \beta x_2^2 + \gamma x_1 x_2, \quad (A.3) \]

where \( \beta \) and \( \gamma \) are positive coefficients. In matrix form:

\[ \dot{V} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (A.4) \]

which is positive definite for all positive determinants, or

\[ \beta > \frac{\gamma^2}{4}. \quad (A.5) \]

The derivative of (A.3) is

\[ \dot{V} = 2x_1 x_2 + 2\beta x_2 a + \gamma x_1 a + \gamma x_2^2. \quad (A.6) \]

Substituting (A.1) yields the quadratic form

\[ \dot{V} = - \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \gamma k_e & \frac{\gamma k_v}{2} + \beta k_e - 1 \\ \frac{\gamma k_v}{2} + \beta k_e - 1 & 2\beta k_v - \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (A.7) \]
which is always less than zero as long as the matrix is positive definite. This condition is equivalent to the determinant being positive:

$$-\beta^2 k_e^2 - \frac{\gamma^2}{4} k_v^2 + \beta \gamma k_e k_v + (2\beta - \gamma^2) k_e + \gamma k_v - 1 > 0.$$  \hspace{1cm} (A.8)

Using the quadratic formula on $\beta$ shows the range of acceptable values to be:

$$\frac{\gamma k_v + 2}{2k_e} - \frac{1}{k_e} \sqrt{2\gamma k_v - \gamma^2 k_e} < \beta < \frac{\gamma k_v + 2}{2k_e} + \frac{1}{k_e} \sqrt{2\gamma k_v - \gamma^2 k_e},$$  \hspace{1cm} (A.9)

for which solutions only exist if

$$\frac{\gamma}{2} < \frac{k_v}{k_e},$$  \hspace{1cm} (A.10)

and in this case a valid solution is always

$$\beta = \frac{\gamma k_v + 2}{2k_e}.$$  \hspace{1cm} (A.11)

Therefore a quadratic Lyapunov function (A.3) can always be found for any positive values of the gains $k_e$ and $k_v$ which ensures global, asymptotic stability for this system.

**Theorem 7.** The controls (A.1) cause the system (A.2) to be asymptotically stable on the restricted domain

$$x_1 \in \mathbb{R} \text{ and } x_2 \in [-v_{\text{max}}, v_{\text{max}}],$$  \hspace{1cm} (A.12)

even when the controls saturate, i.e.

$$|a| \leq a_{\text{max}}.$$  \hspace{1cm} (A.13)

**Proof.** Using the same Lyapunov function as the previous proof (A.3), recall that its derivative is

$$\dot{V} = 2x_1 x_2 + \gamma x_1 a + 2\beta x_2 a + \gamma x_2^2.$$  \hspace{1cm} (A.14)

To ensure that $\dot{V}$ remains negative even in the worst case, consider $|x_1| \to \infty$. In this case, $|a| = a_{\text{max}}$, $ax_1 < 0$, and the first two terms of (A.14) will dominate the rest:

$$\lim_{|x_1| \to \infty} \dot{V} = 2|x_1 x_2| - \gamma |x_1 a_{\text{max}}|$$

$$\leq 2|x_1||x_2| - \gamma |x_1| a_{\text{max}}$$

$$\leq |x_1|(2v_{\text{max}} - \gamma a_{\text{max}}).$$  \hspace{1cm} (A.15)
To ensure $\dot{V}$ is negative, the following condition must hold:

$$\frac{\gamma}{2} > \frac{v_{\text{max}}}{a_{\text{max}}}.$$  \hspace{1cm} (A.16)

Combining this result with (A.10) shows that the control gains must satisfy

$$\frac{k_u}{k_e} > \frac{v_{\text{max}}}{a_{\text{max}}},$$ \hspace{1cm} (A.17)

in order for there to exist a quadratic Lyapunov function that guarantees asymptotic stability over the whole domain.

Interestingly this result does not prove that the system is unstable for gains outside the given range (since Lyapunov is only a sufficient, not a necessary condition for stability). On the contrary, a simple thought experiment will show that this system should be stable for all positive gains just as in the unsaturated case. However, the lack of existence of a quadratic Lyapunov function may have implications for a loss of robustness for these systems.