Oscillatory Control for Constant-Speed Unicycle-Type Vehicles

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Abstract— The work in this paper addresses a method for relating the dynamics of a constant-speed planar unicycle vehicle to another dynamical model that is less constrained. A constant-speed model is appropriate for Unmanned Aerial Vehicles (UAVs), as their airspeed range is often significantly constrained. An oscillatory regulator is examined that controls a limit-cycle behavior. The regulator is based on the idea of defining a center of oscillation (CO) and using a parameterized oscillatory input to produce desirable CO dynamics. This definition allows the CO to be modeled in its own right, and its dynamics are subject to fewer constraints than those of the original vehicle. The CO can then be controlled using any of a variety of outer-loop controllers. Results are demonstrated in simulation for a particular choice of outer-loop controller.

I. INTRODUCTION

Coordinated control of Unmanned Aerial Vehicles (UAVs) is a relatively new and quickly growing field of study. It presents unique challenges to the control designer because of the underactuated nature of aircraft as well as the physical limitations on the controls. The difference between this type of vehicle and variable speed vehicles is highlighted by the fact that the simple task of maintaining a fixed position is not possible for fixed-wing aircraft.

In fixed-altitude applications where UAVs have a significantly restricted flight envelope, the vehicles are often modeled as planar constant-speed unicycles. A number of techniques have been developed for controlling variable speed vehicles (e.g. [1], [2]), but methods for constant-speed vehicles are fewer (e.g. [3] for path following and [4], [5] for trajectory tracking). Extensions of these ideas to multivehicle systems can be found in [4], [6], [7], but in these applications either the number of vehicles or the allowable types of trajectories are limited.

Oscillatory control of nonlinear systems has been studied in a number of instances using either classical averaging theory [8], [9], [10] or methods from differential geometry [1], [2], [11]. However, one of the conditions of averaging theory is that the oscillatory terms must have constant amplitude over whole periods, allowing, at best, discrete-time state feedback. The tools for state feedback from differential geometry, on the other hand, do allow state-valued amplitude modulation of oscillations, but an underlying assumption for existing results is that the systems are small time locally controllable (possibly only configuration) controllable. The work here differs from these techniques in two primary aspects: the system is not small time locally controllable (although it is controllable), and the amplitude of the oscillatory terms will be allowed to vary freely as a function of the state.

The methods developed in this paper are based on the idea of a “Center of Oscillation” (CO), which is an approximation of the actual dynamics (though not a true average) and has fewer constraints than the actual system. In particular, while the original dynamics are controllable but not small time locally controllable (STLC), the CO dynamics are STLC, thus allowing the use of a broader range of techniques for the CO.

The paper is organized as follows. In Section II the control objective and the basic vehicle model are given. The oscillatory inputs as well as the CO and its model are given in Section III. An example outer-loop controller is shown in Section IV, along with simulations. Concluding remarks and topics of current and future work are given in Section V.

II. PROBLEM FORMULATION

Although full aircraft dynamics are quite complex, the essential components for level cruise flight can be captured by the model of a constant-speed unicycle

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ \psi \\ \bar{v} \end{bmatrix} = \begin{bmatrix} v \cos \psi \\ v \sin \psi \\ u \end{bmatrix},$$

(1)

where \( v \) is the (constant) speed of the vehicle, and \( u \) is the control input (heading rate). This model is challenging to control because the vehicle cannot stop (nor move directly sideways), and so it is not STLC. The purpose of this paper is to develop input and state maps that relate this system to a system that is STLC and has dynamics

$$\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\psi} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \bar{v} \cos \bar{\psi} - u_1 \sin \bar{\psi} \\ \bar{v} \sin \bar{\psi} + u_1 \cos \bar{\psi} \\ u_2 \\ u_3 \end{bmatrix},$$

(2)

where \( u_1, u_2, \) and \( u_3 \) are control inputs and \( u_1 \) will be designed to only be available when \( \bar{v} \) is small. Therefore at high speed this model will be that of a unicycle with acceleration input:

$$\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\psi} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \bar{v} \cos \bar{\psi} \\ \bar{v} \sin \bar{\psi} \\ u_2 \\ u_3 \end{bmatrix}.$$

(3)

Note that this system is also STLC and has an equilibrium at the origin. At low speed and with a transformation to
body-fixed coordinates, the model (2) becomes that of a fully controllable linear system:

\[
\frac{d}{dt} \begin{bmatrix} \bar{x}_b \\ \bar{y}_b \\ \bar{\psi} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.
\]  (4)

Once input and state maps are derived, it will be shown that the distance between the positions of the vehicle (1) and this average system is a bounded, symmetric limit cycle where at each half period \( \bar{x} = x \) and \( \bar{y} = y \).

III. OSCILLATORY CONTROL

The limiting condition for (1) that precludes STLC is the lack of arbitrary control of translational velocity. In [7] the average velocity of three such vehicles was controlled by a coupled heading-rate oscillation. Motivated by that approach, consider the system (1) with heading rate parameterized by amplitude and fixed frequency:

\[
u(t) = A(t) \cos(\omega t).
\]  (5)

When \( A \) is constant, the vehicle effectively slows its forward progress by switching back across its path. When \( A = 0 \) the vehicle travels at full speed, \( v \), but as \( A \) increases, the effective vehicle progress slows down.

From the standpoint of control and trajectory tracking, one is now more interested in the mean path of the vehicle, rather than the vehicle’s actual trajectory. As will be shown for a vehicle model (1) with input (5), this mean path can be modeled by (2), which will be referred to as the center of oscillation (CO). In this model \( \bar{v} \) is the effective speed of net progress of the vehicle, \( \bar{\psi} \) is the orientation of this effective motion, and \((\bar{x}, \bar{y})\) denote the position on the mean path and will equal \((x, y)\) at half period intervals. The relationship between the vehicle and the CO (for \( \bar{\psi} = 0 \)) is shown graphically in Fig. 1 for three different values of \( \bar{v} \). The exact relationship between \( A \) and \( \bar{v} \) is discussed below.

For generality, control of the phase and frequency of the oscillation is also desirable, as such control allows multiple vehicles to be synchronized. To this end, the control (5) can be generalized to the form

\[
u(t) = A(t) \cos(\phi(t)),
\]  (6)

where

\[
\dot{\phi}(t) = \omega(t),
\]  (7)

and \( \omega \) is a control variable. Now consider a simple first-order controller for \( \phi \):

\[
\omega(t) = \omega_0 - k_\omega(\phi(t) - \omega_0 t - \phi_0),
\]  (8)

where \( \omega_0 \) is the trim frequency, \( k_\omega \) is a positive gain, and \( \phi_0 \) is the commanded reference phase. The phase error, \( (\phi - \omega_0 t - \phi_0) \), is wrapped to the domain \((-\pi, \pi]\) and will converge to zero with this controller. To facilitate tractable analysis in the following work, \( \omega \) will be chosen to have limited range and small rate. Choosing \( k_\omega \) such that \( k_\omega \pi << \omega_0 \) ensures these constraints are met.

In the preceding, only straight motions of the CO were allowed. To enable a full range of motion, let

\[
u = A \cos(\phi) + B + C \sin(2\phi).
\]  (9)

In this formulation, \( A, B, \) and \( C \) are each inputs. However, in modeling the CO dynamics, \( A, B, \) and \( C \) are assumed to be slowly-varying to simplify analysis. As long as the gains are chosen such that the frequencies of the outer-loop controller are significantly lower than \( \omega \), then approximating the inputs as having zero derivative is reasonable. The exact bound on “significant” has yet to be determined (an order of magnitude is conservative), but the simulations shown in Section IV demonstrate appropriate choices of gains.

With slowly-varying inputs, the \( A \) and \( C \) terms of (9) average to approximately zero. Therefore the bias, \( B \), has primary effect on the overall turning of the system, or more formally, \( \dot{\psi} \approx B \). This effect is demonstrated in Fig. 2. With \( A \) controlling \( \bar{v} \), and \( B \) controlling its heading rate, the CO now appears to be a variable-speed unicycle similar to (3):

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\psi} \end{bmatrix} \approx \begin{bmatrix} \bar{v} \cos \bar{\psi} \\ \bar{v} \sin \bar{\psi} \\ B \end{bmatrix},
\]  (10)

which is a vast improvement in maneuverability over the constant-speed nature of the original vehicle.

The effect of the \( C \) term on the CO dynamics is less straightforward. First, it only has a meaningful effect when \( \bar{v} \) is near zero, because it is designed to take advantage of the nonlinear coupling between the \( A \) term and the \( C \) term. As shown in Fig. 2, the effect of \( C \) is to give the system (10) velocity in a direction perpendicular to the heading of the CO. The CO model emulates (2):

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\psi} \end{bmatrix} \approx \begin{bmatrix} \bar{v} \cos \bar{\psi} - rC \sin \bar{\psi} \\ \bar{v} \sin \bar{\psi} + rC \cos \bar{\psi} \\ B \end{bmatrix},
\]  (11)

which is linear and controllable when transformed to body-fixed coordinates, as in (4). There is a proportionality constant, \( r \), which relates \( C \) to \( u_3 \) from (2).

In order to find relations that are independent of \( v \) and \( \omega \), it is useful to construct a few parameters. First are normalized forms of \( \bar{v} \) and \( A \):

\[
\nu \equiv \frac{\bar{v}}{v} \quad \text{and} \quad \eta \equiv \frac{A}{\omega},
\]  (12)
where \( \nu < 1 \) is termed the speed fraction and \( \eta \) the oscillation parameter. Last is a scale parameter,
\[
\Upsilon \equiv \frac{v}{\omega},
\]
which has units of length and is used to normalize any distances in the system formulation. These parameters will be used extensively in the following analysis and allow the control to be written as
\[
u = \eta \omega \cos(\phi) + B + C \sin(2\phi).
\]

A. Mapping the Vehicle to the Center of Oscillation

Although the CO has its own dynamic model, in order to use the CO to construct control for (1), each CO state is in fact determined entirely by the current state of the vehicle and input. In order to formulate this mapping in a tractable way, approximate steady state behavior is assumed for the system, meaning the CO moves in a constant direction with a constant speed. This assumption implies \( B \approx C \approx 0 \), and \( \eta \) and \( \omega \) are constant (or slowly-varying).

Beginning with the definition of vehicle heading, and substituting (6) into the definition of \( \psi \) from (1) yields
\[
\psi(t) = \int_0^t u(\tau)d\tau = \int_0^t \eta \omega \cos(\phi(\tau))d\tau.
\]
Performing the integration while keeping in mind that \( \dot{\phi} = \omega \approx \omega_0 \) gives
\[
\psi \approx \eta \sin(\phi) + \tilde{\psi},
\]
where \( \tilde{\psi} \) is the integration constant for \( \psi \) as well as the heading of the CO. Thus
\[
\tilde{\psi} \approx \psi - \eta \sin(\phi)
\]
defines the heading for the CO kinematics, such that in steady-state, \( \psi \) oscillates about the constant \( \tilde{\psi} \).

Returning to the relationship between \( \eta \) and \( \nu \), finding the speed of the CO requires integrating the vehicle’s motion over a whole cycle, then dividing by the period. Because the vehicle’s net motion is in the direction of \( \psi \), one can (without loss of generality) simply integrate \( \dot{x} \) with \( \psi = 0 \) to obtain
\[
\dot{\psi} = \frac{1}{T} \int_0^T \nu \cos(\psi)dt,
\]
where \( T = 2\pi/\omega \). Substituting (16) for \( \psi \) and changing the integration variable from time to phase yields
\[
\nu = \frac{1}{2\pi} \int_0^{2\pi} \cos(\eta \sin(\phi))d\phi,
\]
whose solution is a Bessel function of the first kind, of order zero:
\[
\nu = J_0(\eta).
\]
The function, \( J_0 \), is shown in Fig. 3 for reference. The value \( \nu = 0 \) occurs at \( \eta \approx 2.40 \), which means that for this value the vehicle will trace a figure-eight that will not move (see Fig. 1). This state corresponds to the vehicle turning back and forth exactly enough to close its path. In this work the domain of \( J_0 \) is restricted to \( \eta \in [0,3] \), resulting in the range \( \nu \in [-0.26,1] \). Over this domain, \( J_0 \) is invertible, and negative \( \nu \) values result in the ability for the CO to reverse.

To find the position of the CO, first construct a relative position vector between the CO and the vehicle, referenced to the CO’s heading:
\[
\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \equiv \begin{bmatrix} \cos(\tilde{\psi}) & \sin(\tilde{\psi}) \\ -\sin(\tilde{\psi}) & \cos(\tilde{\psi}) \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix}.
\]
With the controls (14), both \( \hat{x} \) and \( \hat{y} \) will oscillate (see Fig. 1). The exact shape of these oscillations cannot be found in closed form, however numerical approximations can be made so that the CO position can be used for feedback. The \( \hat{x} \) and \( \hat{y} \) oscillations are found by running simulations of the vehicle over one period and comparing the result to the mean path definition of the CO. These oscillations (normalized by the scale parameter \( \Upsilon \)) can be characterized using a Discrete Fourier Transform over the range of \( \eta \) values. A sample of this analysis for \( \eta = 2.4 \) is shown in Fig. 4. This result shows that the first two dominant terms of the Fourier series make a good approximation of the true motion:
\[
\begin{align*}
\hat{x} &\approx \Upsilon \left[g_2(\eta) \sin(2\phi) + g_4(\eta) \sin(4\phi)\right] \\
\hat{y} &\approx -\Upsilon \left[h_1(\eta) \cos(\phi) + h_3(\eta) \cos(3\phi)\right],
\end{align*}
\]
where \( g_2, g_4, h_1 \) and \( h_3 \) are Fourier magnitudes as functions of \( \eta \). Numerical approximations of these functions are shown in Fig. 5. Finding the actual position of the CO (\( \hat{x}, \hat{y} \)) is
accomplished by simply subtracting the \( \dot{x} \) and \( \dot{y} \) from the vehicle coordinates (within the CO’s reference frame). Fig. 1 uses this method to plot the CO, and it can be seen from the straightness of the CO trajectories that there is little error in this approximation.

B. Filtering

Since the states of the CO depend directly on \( \eta \), using \( \eta \) as a control input could easily result in a feedback resonance that would drive the system unstable. To avoid this problem, \( \eta \) is filtered by an integrator. The new control input, \( a \), acts like an acceleration. The extended vehicle model is then filtered by an integrator. The new control input, \( a \), acts like an acceleration. The extended vehicle model is then

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} x \\ y \\ \psi \\ \phi \\ \eta \end{bmatrix} &= \begin{bmatrix} v \cos \psi \\ v \sin \psi \\ \eta \omega \cos \phi + B + C \sin(2\phi) \\ \omega \\ a \end{bmatrix},
\end{align*}
\]

where \( \omega \) is given by (8), and \( a, B \) and \( C \) are inputs.

To simplify the relationship between \( a \) and \( \bar{v} \), a linear approximation of \( J_0 \) is used:

\[
q = \left. \frac{dv}{d\eta} \right|_{\eta=2.4} = J'_0(2.4) \approx -0.52. \tag{24}
\]

Therefore the dynamics of \( \nu \) can be represented as

\[
\dot{\nu} = v \frac{dv}{dt} = v \frac{dv}{d\eta} \frac{d\eta}{dt} \approx vqa. \tag{25}
\]

The error this approximation introduces into the model is that the actual acceleration will go to zero as \( \nu \to 0 \). Since the effective gain of the system decreases, it should not adversely affect stability, though it will somewhat reduce performance.

C. CO Model

The approximate CO model (11) is now

\[
\frac{d}{dt} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\psi} \\ \ddot{\phi} \\ \ddot{\eta} \end{bmatrix} \approx \begin{bmatrix} \ddot{v} \cos \ddot{\psi} - pTC \sin \ddot{\psi} \\ \ddot{v} \sin \ddot{\psi} + pTC \cos \ddot{\psi} \\ B \\ vqa \end{bmatrix}, \tag{26}
\]

where in order to properly approximate (1), \( p \approx 0.22 \) is a dimensionless proportionality constant (determined numerically) and \( T \) scales \( C \) to the proper units. The inputs to this system are \( a, B, \) and \( C \).

This model leads to two modes of operation based on the limitations imposed on \( C \). If the CO needs to move quickly (more than about a tenth of the vehicle’s speed), then \( C \) will have little effect, and the model will behave as a variable-speed unicycle with acceleration and turning rate as inputs (\( a \) and \( B \)), similar to the system (3). If the CO needs to move slowly (e.g., for point stabilization), then the model will behave as a fully actuated vehicle with inputs of turning rate, acceleration along \( \dot{\psi} \), and velocity perpendicular to \( \psi \) (\( B, a, \) and \( C \), respectively), similar to the system (2).

As a graphical overview of the construction in the preceding sections, Fig. 6 shows a block diagram. This entire system is then approximated by the CO model (26), so that from the control design point of view, the system looks like Fig. 7. This model is based on approximations near the defined steady-state, and is therefore only valid near this steady-state. However, by assuring that the frequencies of any outer-loop controller used are sufficiently slower than \( \omega \) and that the controller used is robust, these approximations will be sufficiently accurate to ensure stability.

Requiring the controller to be relatively slow also means that the \( B \) and \( C \) terms of the input will be significantly smaller than the \( \eta \) term. This means that this system naturally accounts for heading rate limitations of the vehicle because \( \eta < 3 \), and so

\[
|u| < \eta \omega_0 < 3\omega_0 \tag{27}
\]

for small \( B, C, \) and \( k_{\alpha} \). Therefore if a vehicle has a maximum heading rate, \( \psi_{\text{max}} \), the trim frequency can be chosen as

\[
\omega_0 < \frac{\psi_{\text{max}}}{3} \tag{28}
\]

to ensure no clipping of the commanded control input.
Fig. 6. Block diagram representation of the system approximated by the CO model (26). The dotted box represents the extended vehicle model (23).

Fig. 7. System block diagram used for designing outer-loop controller. The CO model masks the true dynamics shown in Fig. 6.

IV. OUTER-LOOP CONTROL

Given the preceding approximation of the true dynamics, any outer-loop controller can be wrapped around this CO model to achieve desired stability or tracking. Much literature already exists on controllers that are designed for variable-speed unicycles (like mobile robots), for everything from formation keeping to target tracking. Many of these controllers can be applied to this CO model (26), hence expanding their use to aircraft and other systems modeled by (1). One such controller (for point stabilization) is developed below.

Theorem 1: The controls,

\[
\begin{bmatrix}
  a \\
  B \\
  C
\end{bmatrix} = \begin{bmatrix}
  -k_x \bar{v} (\bar{x} \cos \bar{\psi} + \bar{y} \sin \bar{\psi}) - k_v \bar{\psi} \\
  -k_\psi \bar{\psi} \\
  -k_y \bar{\psi} (\bar{y} \cos \bar{\psi} - \bar{x} \sin \bar{\psi})
\end{bmatrix},
\]

applied to the dynamics (26) with all gains positive will globally asymptotically stabilize the CO to the origin.

Proof: Consider the following radially unbounded Lyapunov function,

\[
V = \frac{1}{2} \bar{x}^2 + \frac{1}{2} \bar{y}^2 + \frac{1}{2} \bar{\psi}^2 + \frac{1}{2k_x} \bar{v}^2,
\]

which has derivative

\[
\dot{V} = -k_y (\bar{x} \sin \bar{\psi} - \bar{y} \cos \bar{\psi})^2 - k_\psi \bar{\psi}^2 - \frac{k_v}{k_x} \bar{v}^2.
\]

\(\dot{V}\) is negative definite with respect to \(\bar{y}, \bar{\psi},\) and \(\bar{v},\) but only negative semi-definite with respect to \(\bar{x}.\) Therefore \(\bar{y}, \bar{\psi},\) and \(\bar{v}\) are globally asymptotically stable to the origin. To show that \(\bar{x}\) also converges to zero, consider the system dynamics with \(\bar{\psi} = 0:\)

\[
\frac{d}{dt} \begin{bmatrix}
  \bar{x} \\
  \bar{v}
\end{bmatrix} = \begin{bmatrix}
  -k_x \bar{x} - k_v \bar{v}
\end{bmatrix}.
\]

Because this system is second-order linear with negative eigenvalues for all positive gains, \(\bar{x}\) will also be asymptotically stable to the origin. Therefore the whole system is globally asymptotically stable to the origin.

The following simulations are shown for the previous outer-loop controller (29) as well as a target tracking algorithm (not described in this paper, but similar to others in the literature). These simulations are intended as examples of what is possible, while also validating the frequency separation arguments used earlier to model this system. All of the following simulations use \(v = 1\) and \(\omega = 1.\)

The first simulation is for the controller (29). The vehicle motion is shown in Fig. 8, and the corresponding controls are shown in Fig. 9. As can be seen, the vehicle stabilizes its CO to the proper position and orientation, corresponding to a figure-eight limit cycle motion of the actual vehicle.

The next two simulations use a target tracking outer-loop controller. The first is shown in Fig. 10, and is designed to exhibit the robustness of the modelling method developed
here. The controller employed only ensures that the tracking error decreases to zero if the target’s velocity is constant. Therefore the target is shown at constant velocity, turning at 0.1 rad/s, and oscillating at frequency $\omega$ (all at speed 0.3). While non-constant target velocity does lead to steady-state error, the system still converges to a steady limit cycle that keeps the vehicle near the target.

The final simulation, Fig. 11, shows the effect of phase change and how it can be used in multivehicle coordination. Three vehicles were started with identical initial conditions, but were commanded to change their phase angles to be spaced evenly around the unit circle. As demonstrated, they converge successfully while tracking the same target (which moves at speed 0.2). This method can be useful for getting frequent flyovers of a target.

V. CONCLUSION

In this paper a method has been presented for relating the model of a constant-speed unicycle to that of an STLC system (the CO). This control method keeps the vehicle in a strict pattern around its CO, so that the maximum distance it will deviate is always known (22). In addition, these controls guarantee that the vehicle will fly directly over its CO two times per cycle. The approximations that were made in this construction were validated by simulations.

Much work remains to be done with this method of control, but the results thus far are promising. The techniques presented here are particularly applicable to aircraft, where airspeeds are often restricted to the point that modeling them as constant-speed is quite reasonable (especially because an efficient cruise speed can be chosen). Additionally, the methods place natural limits on maximum turn rate. Therefore the frequency, $\omega_0$, can easily be chosen small enough that the controls never command a turning rate that is out of the vehicle’s range.

Topics for continuing and future work include testing this control technique on an actual aircraft, and thereby showing its robustness to real-world difficulties. Further, the results here will be made concrete with respect to classical averaging theory and differential geometric techniques. Finally, strict bounds will be placed on the allowable frequency of any outer-loop controller.

REFERENCES